CLASS -11

SUBJECT- MATHEMATICS

BINOMIAL THEOREM (द्विपद प्रमेय)

- **BINOMIAL EXPRESSION-**

An algebraic expression containing two terms is called a binomial expression. For example: (x+y), (3a-2b),$(\frac{5}{x}+\frac{1}{y})$ etc. are binomial expressions.

- **TRINOMIAL EXPRESSION:**

An algebraic expression containing three terms is called a trinomial expression. For example : (a+b+c), (2x+3y+z),(x-y-5z),(x-1/y+3/z),etc. are trinomial expressions.

NOTE: In general, expressions containing more than two terms are known as multinomial expressions.

NOTE :The general form of the binomial expression is (a + b) and the expansion of (a + b)^n ,n $\in$ N an is called the binomial theorem. This theorem was first given by Sir Isaac Newton It gives a formula for the expansion of the powers of a binomial expression.

- $(a+b)^0=1$
- $(a+b)^1=a+b$
- $(a+b)^2=a^2+2ab+b^2$
- $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
- $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$
- $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
- $(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab5+b^6$

PASCAL’S TRIANGLE (OR MERU PRASTARA BY PINGLA):

INDEX OF BINOMIAL | **COEFFICIENTS OF VARIOUS TERMS**

0								1						
1							1		1					
2						1		2		1				
3					1		3		3		1			
4				1		4		6		4		1		
5			1		5		10		10		5		1	
6		1		6		15		20		15		6		1

INDEX OF BINOMIAL **COEFFICIENTS OF VARIOUS TERMS**

0				${}^{0}C_{0}=1$			
1			${}^{1}C_{0}=1$		${}^{1}C_{1}=1$		
2			${}^{2}C_{0}=1$	${}^{2}C_{1}=2$	${}^{2}C_{2}=1$		
3		${}^{3}C_{0}=1$	${}^{3}C_{1}=3$		${}^{3}C_{2}=3$	${}^{3}C_{3}=1$	
4		${}^{4}C_{0}=1$	${}^{4}C_{1}=4$	${}^{4}C_{2}=6$	${}^{4}C_{3}=4$	${}^{4}C_{4}=1$	
5	${}^{5}C_{0}=1$	${}^{5}C_{1}=5$	${}^{5}C_{2}=10$		${}^{5}C_{3}=10$	${}^{5}C_{4}=5$	${}^{5}C_{5}=1$
6	${}^{6}C_{0}=1$	${}^{6}C_{1}=6$	${}^{6}C_{2}=15$	${}^{6}C_{3}=20$	${}^{6}C_{4}=15$	${}^{6}C_{5}=6$	${}^{6}C_{6}=1$

***NOTE:** We know that

$${}^{n}C_{r}= n! / r!\,(n-r)!\,,\quad 0 \le r \le n$$

$${}^{n}C_{0} = 1 = {}^{n}C_{n} \qquad {}^{n}C_{r} = {}^{n}C_{n-r}$$

$${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r} = C\,(n+1, r),\quad 1 \le r \le n$$

$${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x=y \text{ or } x+y=n$$

$${}^{n}C_{r} / {}^{n}C_{r-1} = (n-r+1)/r$$

$${}^{n}C_{r+1} / {}^{n}C_{r} = (n-r)/(r+1)$$

*$(a+b)^0=1$

$(a+b)^1=a+b={}^1C_0\,a+{}^1C_1\,b$

$(a+b)^2=a^2+2ab+b^2={}^2C_0\,a^2+{}^2C_1\,ab+{}^2C_2\,b^2$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3={}^3C_0\,a^3+{}^3C_1\,a^2b+{}^3C_2\,ab^2+{}^3C_3\,b^3$

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4={}^4C_0\,a^4+{}^4C_1\,a^3b+{}^4C_2\,a^2b^2+{}^4C_3\,ab^3+{}^4C_4\,b^4$

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

$={}^5C_0\,a^5+{}^5C_1\,a^4b+{}^5C_2\,a^3b^2+{}^5C_3\,a^2b^3+{}^5C_4\,ab^4+{}^5C_5\,b^5$

$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$

$={}^6C_0\,a^6+{}^6C_1\,a^5b+{}^6C_2\,a^4b^2+{}^6C_3\,a^3b^3+{}^6C_4\,a^2b^4+{}^6C_5\,ab^5+{}^6C_6\,b^6$

$(a+b)^7={}^7C_0\,a^7+{}^7C_1a^6b+{}^7C_2\,a^5b+{}^7C_3\,a^4b^3+{}^7C_4\,a^3b^4+{}^7C_5a^2b^5+{}^7C_6\,ab^6+{}^7C_7\,b^7$

$=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

*BINOMIAL THEOREM FOR ANY POSITIVE INTEGER n :

$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}\, b^2 +$

${}^nC_3\, a^{n-3}b^3 + \ldots\ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_n b^n$

Proof- Let the given statement be

$P(n): (a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}\, b^2 +$

${}^nC_3\, a^{n-3}b^3 + \ldots\ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_n b^n$

For n=1,we have

$P(1): (a+b)^1 = {}^1C_0\, a + {}^1C_1\, b = a+b$ (Using ${}^1C_0 = 1 = {}^1C_1$)

Thus, P(1) is true.

Suppose P(k) is true for some positive integer k, i.e.

$(a+b)^k = {}^kC_0\, a^k + {}^kC_1\, a^{k-1}b + {}^kC_2\, a^{k-2}\, b^2 +$

${}^kC_3\, a^{k-3}b^3 + \ldots\ldots + {}^kC_{k-1}ab^{k-1} + {}^kC_k\, b^k$ ....(1)

We shall prove that P(k+1) is also true,i.r.,

$(a+b)^{k+1} = {}^{k+1}C_0\, a^{k+1} + {}^{k+1}C_1\, a^k b + {}^{k+1}C_2\, a^{k-1}\, b^2 +$

$+ \ldots\ldots + {}^{k+1}C_k\, ab^k + {}^{k+1}C_{k+1}\, b^{k+1}$

Now, $(a+b)^{k+1} = (a+b)(a+b)^k$

$= (a+b)\,({}^kC_0\, a^k + {}^kC_1\, a^{k-1}b + {}^kC_2\, a^{k-2}\, b^2 + {}^kC_3\, a^{k-3}b^3 +$
$\ldots\ldots + {}^kC_{k-1}ab^{k-1} + {}^kC_k\, b^k)$ [from 1]

$= {}^kC_0\, a^{k+1} + {}^kC_1\, a^k b + {}^kC_2\, a^{k-1}\, b^2 + {}^kC_3\, a^{k-2}\, b^3 + \ldots$

$\ldots\ldots + {}^kC_{k-1}a^2b^{k-1} + {}^kC_k ab^k + {}^kC_0\, a^k\, b + {}^kC_1\, a^{k-1}\, b^2 +$

${}^kC_2\, a^{k-2}\, b^3 + {}^kC_3\, a^{k-3}b^4 + \ldots\ldots + {}^kC_{k-1}ab^k + {}^kC_k\, b^{k+1}$

$= {}^kC_0\, a^{k+1} + ({}^kC_1\, a^k b + {}^kC_0\, a^k\, b) +$

$({}^kC_2\, a^{k-1}\, b^2 + {}^kC_1\, a^{k-1}\, b^2) +$

$({}^kC_3\, a^{k-2}\, b^3 + {}^kC_2\, a^{k-2}\, b^3) + \ldots\ldots.$

$\ldots + ({}^kC_k ab^k + {}^kC_{k-1} ab^k) + {}^kC_k\, b^{k+1}$

$= {}^kC_0\, a^{k+1} + ({}^kC_1 + {}^kC_0)\, a^k b +$

$({}^kC_2 + {}^kC_1)\, a^{k-1}\, b^2 +$

$({}^kC_3 + {}^kC_2)\, a^{k-2}\, b^3 + \ldots\ldots.$

$\ldots + ({}^kC_k + {}^kC_{k-1})\, ab^k + {}^kC_k\, b^{k+1}$

$= {}^{k+1}C_0\, a^{k+1} + {}^{k+1}C_1\, a^k b + {}^{k+1}C_2\, a^{k-1}\, b^2 +$

$+ \ldots\ldots + {}^{k+1}C_k\, ab^k + {}^{k+1}C_{k+1}\, b^{k+1}$

(by using ${}^{k+1}C_0 = {}^kC_0 = 1$, ${}^kC_1 + {}^kC_0 = {}^{k+1}C_1$, ${}^kC_2 + {}^kC_1 = {}^{k+1}C_2$, ${}^kC_3 + {}^kC_2 = {}^{k+1}C_1, \ldots\ldots\ldots.$

${}^kC_k + {}^kC_{k-1} = {}^{k+1}C_k$ & ${}^kC_k = 1 = {}^{k+1}C_{k+1}$)

Thus, it has been proved that P(k+1) is true whenever P(k) is true.Therefore,by the principle of mathematical induction,P(n) is true for every positive integer n.

NOTE:

(1) $$(a+b)^n=\sum_{k=0}^{n} {}^nC_k\, a^{n-k}\, b^k$$

$$(a+b)^n= {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b+ {}^nC_2\, a^{n-2}\, b^2+ {}^nC_3\, a^{n-3}b^3 + \ldots\ldots+{}^nC_{n-1}ab^{n-1}+ {}^nC_nb^n$$

(2) **The coefficients of terms equidistant from beginning and end are equal.**

($^nC_0 = {}^nC_n,\ {}^nC_1 ={}^nC_{n-1},\ {}^nC_2 ={}^nC_{n-2}$,........)

(3) **^nC_r is the binomial coefficient in the expansion of $(a+b)^n$.**

(4) **There are (n+1) terms in the expansion of $(a+b)^n$.**

(5) **In the successive terms of the expansion the index of a goes on decreasing by unity,starting with n in the first term and ending with n in the last term.**

(6) **In the expansion of the sum of the indices of a and b is N in every term of the expansion.**

***Some special cases:**

$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots\ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_n b^n$

(1) $(x-y)^n = \{x+(-y)\}^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}(-y) + {}^nC_2\, x^{n-2}\,(-y)^2 + {}^nC_3\, x^{n-3}(-y)^3 + \ldots\ldots$

$\ldots\ldots + {}^nC_{n-1}x\,(-y)^{n-1} + {}^nC_n(-y)^n$

$= {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 - {}^nC_3\, x^{n-3}y^3 + \ldots\ldots + (-1)^{n-1}\, {}^nC_{n-1}xy^{n-1} + (-1)^n\, {}^nC_n y^n$

(2) $(1+x)^n = {}^nC_0\,(1)^n + {}^nC_1\,(1)^{n-1}x + {}^nC_2\,(1)^{n-2}\, x^2 + {}^nC_3\,(1)^{n-3}\, x^3 + \ldots\ldots + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots$

$\ldots\ldots\ldots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

(3) $(1+1)^n = 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots$

$\ldots\ldots\ldots + {}^nC_{n-1} + {}^nC_n$

(4) $(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots\ldots$

$\ldots\ldots\ldots + (-1)^{n-1}\, {}^nC_{n-1} x^{n-1} + (-1)^n\, {}^nC_n x^n$

(5) $(1-1)^n = 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots$

$\ldots\ldots\ldots + (-1)^{n-1}\, {}^nC_{n-1} + (-1)^n\, {}^nC_n$

(6) $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

(7) $$(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$

(8) The terms in the expansion of $(x-a)^n$ are alternatively positive and negative, the last term is positive or negative according as n (index) is even or odd.

(9) $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\ x^r$$

$$(x+1)^n = \sum_{r=0}^{n} {}^nC_r\ x^{n-r}$$

So,

$$\sum_{r=0}^{n} {}^nC_r\ x^r = \sum_{r=0}^{n} {}^nC_r\ x^{n-r}$$

$$\{\because (1+x)^n = (x+1)^n\}$$

(10) The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(11) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

(12) $(a+b)^n + (a-b)^n = 2({}^nC_0\, a^n b^0 + {}^nC_2\, a^{n-2} b^2 + {}^nC_4\, a^{n-4} b^4 + ...)$

(13) $(a+b)^n - (a-b)^n = 2({}^nC_0\, a^n b^0 + {}^nC_3\, a^{n-3} b^3 + {}^nC_5\, a^{n-5} b^5 + ...)$

(14)

EXPANSION	NUMBERS OF TERMS	
	n is odd	n is even
$(a+b)^n + (a-b)^n$	(n+1)/2 terms	$(n/2+1)$ terms
$(a+b)^n - (a-b)^n$	(n+1)/2terms	(n/2) terms
$(a+b)^n$	(n+1) terms	(n+1) terms

Q.1. Prove that

$$\sum_{r=0}^{n} {}^nC_r\, 3^r = 4^n$$

Solution: We have,

$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots\ldots$

$\ldots\ldots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

Putting x=3 on both sides,we get

$$(1+3)^n = \sum_{r=0}^{n} {}^nC_r\, 3^r$$

or, $(4)^n = \sum_{r=0}^{n} {}^nC_r\, 3^r$